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MATHEMATICAL NATURALISM AND THE POWERS OF SYMBOLISMS

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ABSTRACT: Advances in modern mathematics indicate that progress in this field of knowledge depends mainly on culturally inflected imaginative intuitions, or intuitive imaginings—which mysteriously result in the growth of systems of symbolism that are often efficacious, although fallible and very likely evolutionary. Thus the idea that a trouble-free epistemology can be constructed out of an intuition-free mathematical naturalism would seem to be question begging of a very high order. I illustrate the point by examining Philip Kitcher's attempt to frame an empiricist philosophy of mathematics, which he calls "mathematical naturalism," wherein he proposes to explain novelty in mathematics by means of the notion of 'rational interpractice transitions,' only to end with an appeal to science to supply a meaning for rationality. A more promising naturalistic approach is adumbrated by Noam Chomsky who begins with a straightforward acceptance of mind and language as 'natural' or concrete facts which bespeak the need for a linguistic faculty. This indicates in turn that there may also be a mathematical faculty capable of generating and exploiting the powers of mathematical symbolisms in a manner analogous to the linguistic faculty.

KEYWORDS: Naturalism; Perception; Epistemic Aims; Retroduction; Rationality; Intuitions; Faculty; Imagination; Symbolism

> "Let the dead bury the dead, but do you preserve your human nature, the depth of which was never yet fathomed by a philosophy made up of notions and mere logical entities."[1]

MATHEMATICS AND 'MINDING'

The question of what mathematics really *is* haunts every philosophy of mathematics. This perennial puzzle is bound up with the question of what mathematics actually contributes to the modern quest for genuine understanding of the world. More specifically, if it is acknowledged that at least some theories of mathematics actually throw important light on natural physical events, the philosopher of mathematics seems bound sooner or later to wonder about the power of symbolisms that appear to bear witness to the possibility of forming intimate relationships between minds and nature. But to get an idea of the range of problems needing to be addressed in this line of

[1] Samuel Taylor Coleridge, *Biographica Literaria: or Biographical Sketches of My Literary Life and Opinions*, ed. Geo. Watson (London: J. M. Dent, 1975), 149.

thought, one need only consider Descartes' fundamental claim that mathematics "is a more powerful instrument of knowledge than any other that has been bequeathed to us by human agency, as being the source of all others" in the same light as the claim that "the human mind has in it something that we may call divine, wherein are scattered the first germs of useful modes of thought."[2]

The irony is that Descartes also indicates the impossibility of giving an adequate account of the foundations of mathematical, or indeed any other type of knowledge, without needing to bring in, perhaps sooner rather than later, spiritual forces or powers that operate beneath the surface of thought. He observes, for instance, that "that power by which we are properly said to know things, is purely spiritual " (HR, 38)." Hence Descartes himself raises doubts about whether the mental powers represented by mathematical symbolisms give the lie to his vision of a philosophy rendered into something like a scientific discipline. Why think that these powers do not present the chief obstacles to understanding cognition *tout court*, a question which puts the very idea of a distinctive epistemology of mathematics into question. He may even be suspected of turning a bad beginning into an absurdity when he insists on detaching the best of his 'mindings' from the contributions of his deceiving body. He is thereby obliged to distort his own experiencing. He maintains, for instance, that he is "not more" than "a thing which thinks"; that is, a thing "which doubts, understands, [conceives], affirms, denies, wills, refuses, which also imagines and feels" (see "Of the Nature of the Human Mind," HR, 153). But although he appears to include feelings along with imagination in this list of important aspects of experiencing, he at the same time denies their relevance even as he tacitly acknowledges their indispensability while pursuing his famous method of doubt. For he proposes to search and sift all his assumptions and "to reject as absolutely false everything as to which I could imagine the least ground of doubt, in order to see if afterwards there remained anything in my belief that was entirely certain" (HR, 101). Yet only a certain *feeling* of doubt is surely able to inform him that he has run up against something blocking his unequivocal and immediate assent.

In brief, it is not logical flaws in carefully articulated quasi-mathematical proofs that cause or signal (to use Peirce's apt phrase) 'genuine doubts'. For although Descartes may be quite right to tie the mental activity of imagining quite closely to the bodily passions ("in imagining [mind] turns towards the body"), he is arguably wildly wrong, along with a good many other modernist philosophers, in what he makes of this turning:

> the power of imagination which is in one, inasmuch as it differs from the power of understanding, is in no wise a necessary element in my nature, or in [my

[2] See "Rules for the Direction of the Mind," *The Philosophical Works of Descartes*, Vol. I, trans. Elizabeth S. Haldane and G.R.T. Ross (Cambridge: Cambridge University Press, 1972), 10-11 (hereafter referred to as HR). Descartes also claims that the best of mathematics bears witness to "spontaneous fruit" which has sprung from "inborn principles" and that "this is the chief result which I have had in view in writing this treatise".

> essence, that is to say, in] the essence of my mind (HR, 186).

For what except imagination could lead anyone to think there is such a thing as an essence of mind, never mind pure, certain and absolutely secure knowledge? And what but imagination could induce anyone to think that there are elements in our very natures that are necessary?

Put yet another way, Descartes both prompts and immediately suppresses a number of important but difficult questions that bear directly on what to make of the idea of mathematical knowledge—of whether or not there really are semi-divine powers standing behind certain acts of minding. For it is possible that all modes of thought enlist (if only tacitly) spiritual powers or agencies that produce what are often referred to as intuitions or insights. This possibility has been reinforced, ironically enough, by the adventures of twentieth century philosophers of mathematics who made many highly ingenious attempts to identify precisely the central core of mathematics without having to make any appeal to intuitions. Their fond hope was to emulate nineteenth century mathematicians who had succeeded, at least on the face of it, in making their proofs in geometry and arithmetic completely rigorous and utterly independent of intuition. The chief participants in this competition believed that success was just around the corner when, in the early twentieth century, they were suddenly confronted by certain paradoxes in formal logic and set theory. Thus the hope of finally laying a foundation for all reasoning in a purified mathematics, in accordance with universal standards of rigor no less exacting than those employed in mathematics itself, achieved just the opposite of its aims.

For betrayed by intractable paradoxes, the *raison d'être* of philosophy of mathematics as an autonomous branch of philosophy began to evaporate. No longer could it be honestly proclaimed as a self-evident truth that mathematical thinking represents the epitome of rational thinking, or that questions about mathematical existence, truth, necessity, and certainty can be quarantined from disputes concerning the meaning of pure, let alone certain and objective, knowledge that were raging in the so-called humanistic areas of philosophical inquiry. One may therefore wonder whether the successors of Descartes afford particularly clear examples of a typically modern disease of the understanding whose symptoms include, among other things, a distorted form of reason and a self-mutilating conception of experience. For Descartes' way of beginning to do philosophy has turned out to be remarkably popular—witness the bulk of twentieth century philosophy of mathematics and science, and especially the combined efforts of those legions of self-styled naturalists who focus their attentions not on nature but rather on the deliverances of science. On the basis of this supposedly rational decision they then proceed to tackle deep philosophical problems having general epistemological and ontological significance.

Yet what could be a more reasonable and empirically well-grounded way to approach Nature than by first noting that a 'thinking thing' requires a functioning brain which in company with other organs of the body sometimes produces (who knows how?) those moments of sensibility that keep the idea of knowledge afloat. For a brain is but one organ of a sentient warm-blooded feeling body. And there is no evidence whatsoever that cold unfeeling corpses are capable of making assertions like that of Descartes: "I know for certain that nothing of all that I can understand by means of my imagination belongs to this knowledge which I have of myself, and that it is necessary to recall the mind from this mode of thought with the utmost diligence in order that it may be able to know its own nature with perfect distinctness" (HR, 152-53). Indeed, few philosophical presuppositions may be more deserving of close scrutiny than Descartes' confident declaration that "we should busy ourselves with no object about which we cannot attain a certitude equal to that of the demonstrations of Arithmetic and Geometry."(HR, 5) Everyday experience suggests that nothing *except* bodily informed, affectively controlled, acts of intuitive imagination, or imaginative intuition, have the power to grasp anything worth hanging onto at all.

Descartes, in short, may have given voice not just to a private and eccentric assumption about how to begin to do philosophy but rather may have inadvertently provided cause to suspect this culture of encouraging badly confused modes of thought that are prone to embrace prejudices that are founded neither in reason nor experience. Yet whenever philosophy of mathematics is pursued as a separate discipline, or as a branch of philosophy of science, the result is not, as one might have expected, an upsurge of calls for a complete reassessment of the Cartesian view of mathematical epistemology, which might well include a suggestion that philosophers of mathematics need to begin all over again, this time taking the possibility of intuitive imaginings, or imaginative intuitions, more seriously. On the contrary, the failure of the foundational programmes in philosophy of mathematics appears to have driven the topics of intuition and imagination even deeper underground. For intuitions never came close to being eliminated from the three main foundationalist programmes in philosophy of mathematics (logicism, formalism, and intuitionism), since here each worker sought to find key or 'basal intuitions' (to use Brouwer's term).

In brief, then, the failure of foundationalism does not affect in any important way the question of the existence, or otherwise, of mathematical intuitions. Indeed, the relevance of the intuitive/imaginative aspects of mathematical creation have become ever more undeniable as traditional beliefs about mathematics have one by one collapsed—such as that mathematics is the repository of precise, other-worldly truths; that the aims and methods of mathematicians are consciously shaped by deductive thought processes and well-defined methods of discovery and proof; that mathematical knowledge, unlike other types of knowledge, is cumulative and linearly progressive in its development.[3] But to follow through with this collapse is eventually to find oneself entertaining radical doubts not only about the universality of mathematical truth, it is

[3] For a discussion of these and other received views of mathematics, see Michael J. Crowe, "Ten Misconceptions About Mathematics," in William Aspray and Philip Kitcher, eds., *History and Philosophy of Modern Mathematics* (Minneapolis: University of Minnesota Press, 1988), 260- 75 (hereafter referred to as MN).

ultimately to put into question the idea that there is a separate discipline called philosophy of mathematics. Indeed, one may begin to wonder whether there is even such a thing as philosophy of science in so far as this derives its own *raison d'être* from a Cartesian belief in the unique cognitive powers of mathematics. One may even end up wondering whether there is any significant difference between mathematical and poetical modes of knowing. It is for reasons such as these that it seems worthwhile to explore the issue of intuitions in mathematics, and the question of what role, if any, imagination should play in attempts to frame a naturalistic philosophy of mathematics, assuming there is such a thing and that it can be entirely liberated from the need to appeal to intuitions.

CAN THERE BE A MATHEMATICAL NATURALISM WITHOUT INTUITIONS?

That this can be done is in fact the principal claim of Philip Kitcher who expressly holds that it is possible to give a principled account of mathematical knowledge without invoking "special procedures"—i.e., "enlightenment by Platonic intuition, construction in pure intuition, stipulative fixing of the meaning of terms, or whatever"—in order to justify knowledge of the axioms of mathematics.[4] Claiming that his version of mathematical naturalism is a more sophisticated and comprehensive version of empiricism than the one usually associated with the name of John Stuart Mill, he maintains that only an overweening and misbegotten desire for an apriorist epistemology has induced modern philosophers of mathematics to reject empiricism as hopelessly simplistic and not a viable alternative to foundationalism. Yet once one has finally overcome the temptation to think there is knowledge independent of experience, "there is no plausible alternative to a naturalistic mathematical epistemology" (MN, 317).

This is not to deny, says Kitcher, that "special moments" do occur in mathematics and sometimes lead to useful results. Such moments are however not epistemologically significant: "Platonic or constructivist intuition, stipulative definition, yield knowledge—to the extent that they function at all—only against the background of a kindly experience that underwrites their deliverances" (MN, 295).

But what sort of 'kindly experience' could underwrite Kitcher's confidence that there are no viable alternatives to his approach? What sort of experience could even indicate that intuitions are *not* 'natural entities'? Is there any reason at all for believing that practicing mathematicians are simply deluded when they claim that intuitions and insights have led them to their unexpected discoveries? Yet Kitcher holds that the esoteric abstractions of modern mathematics can be viewed as an evolved product of changing theories that have developed from roots which do not include special intuitions but rather 'unproblematic entities'—that is, primitive, empirically grounded practices of human beings performing everyday operations on their environment.

[4] See Philip Kitcher, "Mathematical Naturalism" (in MN, 293-325), 294.

Kitcher's answer to the questions posed above turns on the claim that explanations of the emergence of novelty in mathematics can be expressed in terms of sequences of intelligible, orderly changes in mathematical problem-posing and problem-solving, changes that are moreover not merely accidental or conventional. When mathematical practices are transmitted from one generation of workers to another, and are modified by new creative workers in the field, the resulting modifications that survive criticism bear witness to increments of progress in rational understanding. So it is important to note at once that Kitcher's story of the development of mathematics does not (as he thinks) require explicit demonstrations of how to link every kind of esoteric mathematical entity to 'primitive' empirical objects. Indeed, it is extremely doubtful whether it even makes sense to try to link certain hotly defended (by David Hilbert, for instance) 'actually infinite' sets with transfinite cardinality to finite operations performed on the environment. Thus understandably keen to circumvent this sort of conundrum, Kitcher proffers a definition of mathematics that is operational mainly in character: for mathematics is

> an idealized science of human operations. The ultimate subject matter of mathematics is the way in which human beings structure the world, either through performing crude physical manipulations or through operations of thought (MN, 313).

This definition, however, obliges him to look for an empiricist elucidation of the vague notion of "operations of thought" that are currently performed on highly abstract symbolisms and not on objects in the environment. At this point his story begins to sidle away from experience of ordinary things (simple physical manipulations performed on macro-objects) toward questionable assumptions concerning the nature of mental operations. Indeed, Kitcher himself indicates that mathematical naturalism would be better termed "naturalistic constructivism," for it really pivots not on the deliverances of a "kindly experience" but rather on the rationality of certain "interpractice transitions." These transitions provide the links, he maintains, that connect in continuous chains those mathematical 'objects' that have come to be firmly established in the discipline to roots that lie buried somewhere deep in the past. (i.e., in presumably 'primitive' observations related to physical manipulations).

He thereby indicates that his empiricist project must stand or fall on whether he can produce a plausible account of rational interpractice transitions; but this calls for a sound definition of rationality. And here Kitcher fastens on a meaning which he says is endorsed by countless philosophers: being rational "consists in adjustment of means to ends" (MN, 304). Acknowledging, however, that this definition of rationality is not immediately useful in the case of mathematics, since there is no independent notion of mathematical truth that would provide a meaning for 'ends', Kitcher is thus obliged to give his story another dubiously empirical twist by introducing as a pivotal notion the idea of an "epistemic end." Moving ever further away from an empiricism based on "kindly experience," he notes that "the only epistemic end in the case of mathematics is the understanding of the results so far achieved' (MN, 314-15).

Now Kitcher also wants to show that the achievements of mathematicians should not be regarded as all on a par—as simply whatever a certain type of thinker has arrived at in the course of time, where new theoretical developments are subject only to the constraints imposed by existing standards of proof. That is to say, some, but not all, of modern mathematics is worthy of being classified as 'legitimate'. But in order to justify this claim, Kitcher is obliged to link the notion of 'rational interpractice transitions' closely to 'epistemic ends' in the development of mathematics. Briefly, there is room for just one kind of "useless" knowledge in mathematics: those "claims that have in themselves no practical implications but serve to enhance our understanding of results that are practically significant" (MN, 315). As to the question of what *is* practically significant in mathematics his answer hinges on our desire "to bring system and understanding to the physical and mental operations *we find it worth performing* on the objects of our world, so that the shape and content of mathematics are ultimately dictated by our practical interests and the epistemic goals of other sciences" (MN, 315, italics original).

In the end, Kitcher's solution to the problem of what is and is not legitimate or worthwhile preserving in all the creative activity which mathematicians engage in revolves about the claim that the respective problems of the growth of mathematical knowledge and of scientific knowledge run "formally parallel" and so require analogous treatments (MN, 299). But to get here he finds it necessary to subordinate the epistemic aims of mathematicians to those of scientists who aim generally to achieve "greater understanding of some facet of the universe, say of the structure of matter or of the springs of animal behavior" (MN, 305). The upshot is that mathematical and scientific naturalism turn out to be very intimate bedfellows indeed. Mathematical naturalism aspires to an empiricism that seeks its truth not through distinguishing the legitimate elements of mathematics from the illegitimate by explicitly spelling out steps in rational interpractice transitions that connect the useful abstract mathematical objects to physical manipulations on the environment. Rather mathematical naturalism must look to science both for a reason for its existence and for a definition of the rationality of interpractice transitions. To science it presumably returns the favour by providing it with the assurance that a properly constituted scientific empiricism can, at least in principle, be anchored in very ordinary human operations of thought. Yet the exchange of favours can only free itself from the taint of incest if science is indeed capable of providing, as Kitcher promises, "a precise account of rationality and progress in the sciences" (MN, 317).

Thus Kitcher's aim to purge intuitions from naturalistic mathematical epistemology not only fails to show why intuitions do not properly belong to the class of 'natural objects', a class whose membership every empiricist-naturalist must surely want to become clear about (indeed, especially clear since the class of 'scientific objects' has expanded enormously to include a mind-boggling number of highly elusive and evanescent entities which are also the result of "operations of thought" that cannot be directly linked to physical operations). Kitcher's anti-foundationalist aim to show there is no plausible alternative to the sort of mathematical naturalism he espouses turns out in the end to depend upon an undefended faith in the superior rationality of scientific explanations.

The upshot is that mathematical naturalism and scientific naturalism merely take in each other's washing, as it were. Or would it be better to say that, by calling for science to give a precise account of rationality and progress in the sciences, Kitcher also provides a particularly clear illustration of the power of the Myth of Scientific Superrationality to turn otherwise cool heads—for he may have succumbed to the same temptation that seduced Descartes into believing in the privileged status of mathematical knowledge. In any case, Kitcher does not show that appeals to "kindly experience" can be freed from the need to appeal in the end to intuitions or insights as partly informing our understanding of natural events. His notion of rational interpractice transitions merely helps point up the priority of the need to become clearer about one of his principal tenets: "truth is what rational inquiry will produce, in the long run" (MN, 314).

'EPISTEMIC AIMS' OR 'RETRODUCTIVE AIMS'?

At this point the naturalist might well turn for help to C. S. Peirce and his struggles to clarify reason (as in, for instance, his famous essay "How to make our ideas clear") which favour an empiricist approach in which he also tries to do justice at once to science, mathematics, and logic.[5] Peirce holds moreover that it is possible to explain truth and reality as the final opinion that a scientific community of responsible inquirers will arrive at in the long run. He also argues that socially or publicly justified "leading principles or habits" (in contrast to private intuitive insights) ultimately provide the basis of good reasonings.

On the other hand, many of Peirce's most important and penetrating discussions of rational thought and belief point up the indispensability of insights in the 'operations of thought' performed by seekers after truth. For the operations that produce real novelty in scientific or mathematics theories, and which led him to replace the common term 'pragmatism' by 'pragmaticism', call for a distinction in forms of reasoning that is frequently passed over. For Peirce specifically links the proof of pragmaticism to the fact that there are three elementary forms of reasoning, the first of these being abduction which is essential for any kind of theorizing and thus precedes inductions and deductions.[6] Hence pragmaticism refers not to one theory but rather to a whole complex of theories (including Critical Commonsensism which holds that all rational

[5] See, for instance, the introductory essay by Justus Buchler, ed., *Philosophical Writings of Peirce* (New York: Dover Pub. Inc. 1955), esp. p. xv. Buchler notes that in Peirce "we find just recognition alike of the socio-biological and the mathematical aspects of logic"(p. xii).

[6] See C. S. Peirce, *Collected Papers of Charles Sanders Peirce*, vols. I-VI, ed. Charles Hartshorne and Paul Weiss; vol. VII-VIII, ed. Arthur W. Burks (Cambridge, Mass.: Harvard University Press, 1960). References to the Collected Papers will be given in the usual manner; e.g., CP, 8.209.

thought ultimately rests on a (intrinsically unstable) ground of especially vague "acritical indubitable beliefs") that revolve about acts of abduction.

In brief, Peirce holds that not only the production of novelty in science but also any genuine advance in understanding is ultimately dependent on abduction. Yet this movement of mind is intrinsically invisible to sense-oriented empiricists since it depends at bottom on a form of 'guessing'. That is to say, 'abductive inference' is principally a quest for fruitful hypotheses. Peirce thus indicates that a different kind of logic (perhaps one more closely connected to the ancient idea of the *Logos*?) is needed to deal adequately with the idea of a rational explanation, since abduction is not of the same order, strictly speaking, as deduction or induction (experimental reasoning). While the last two modes of inference are required to produce coherent, consistent theories that agree with the empirical evidence, they are always secondary to the framing of hypotheses. Furthermore, the element of guess-work in this side of abduction cannot be brushed aside as *mere* guessing since a great number of possibilities need to be entertained, yet it is frequently the case that only a relatively few guesses are required before the right one is hit upon. Peirce thus suggests that unconscious insights, which are however not infallible, underpin 'right' abductive inferences that lead to novelty not only in science but also in perception itself.[7]

The implication for understanding the allegedly rational transitions that Kitcher believes account for the development of mathematical theories, and which he and Peirce suggest can ultimately lead to the Truth as the common limit of an unlimited number of convergent series of rational inquiries, is that rationality has more to do with the success of abductive 'inferences' than with the number or the nature of the series themselves. Put another way, rationality evokes at once a more temporally 'localized' and ultimately mysterious view of Truth, one that is adumbrated in Peirce's remark that "it is a primary hypothesis underlying all abduction that the human mind is akin to the truth in the sense that in a finite number of guesses it will light upon the correct hypothesis" (CP 7.220).

This postulate of a natural connection between Mind and Nature is consonant with an interpretation of intuitions as perspicuous 'seeings', or right 'seeings-into', even if 'rightness' can only be accounted for in the end by appealing to a consensus within an interested community. If developments in 20th century mathematics have any general epistemological significance it is perhaps just because they provide a dramatic illustration of this point.

That progress in mathematics is often the result of looking backwards in order to try to move forward is well illustrated by the work of Russell and Whitehead in their search for a formal basis in symbolic logic and set theory of arithmetic. In the preface to

[7] Peirce observes, for instance, that "abductive inference shades into perceptual judgment without any sharp line of demarcation between them; or, in other words, our first premises, the perceptual judgments, are to be regarded as an extreme case of abductive inferences, from which they differ in being absolutely beyond criticism. The abductive suggestion comes to us like a flash. It is an act of *insight*, although of extremely fallible insight" (CP 5.181, original emphasis).

Principia Mathematica they note that their logical reconstruction of arithmetic is based on a search for a set of ideas and axioms that are sufficient, but not necessary, to enable them to deduce what they value in ordinary arithmetic. For the most obvious truths of mathematics, they assert, do not reside at the level of the primitive axioms. Only at the level of ordinary arithmetic can the greatest degree of self-evidence be found.[8]

Indeed, it is not difficult to believe that mathematical practice is replete with such retroductive attempts to alter, embellish, or improve extant structures that are selected as embodying valuable 'truths' whose security is nevertheless believed to be in need of improvement. The retroductive nature of modern mathematics is in fact evident from some of Kitcher's examples. He cites Zermelo's systematization of set theory in which Zermelo aimed to save certain widely accepted mathematical ideas that had been tacitly or explicitly employed in reasoning about real numbers. But Zermelo did not *know* that his procedure was the right one, as Kitcher observes, he only proposed that

> these *antecedently accepted* claims could be derived from the principles he accepted as basic. The justification is exactly analogous to that of a scientist who introduces a novel collection of theoretical principles on the grounds that they explain the results achieved by previous workers in the field (MN, 295, my italics).

Indeed, Kitcher appears to base his belief in the superiority of naturalistic over platonistic epistemologies on the grounds that naturalism is better able to take into account such episodes. These generally involve, as he puts it, the assembly of evidence to show that the modification of mathematics through the adoption of the new axiom or concept would bring some advance in mathematical knowledge (MN, 297). It is thus highly significant that Kitcher notes that such retroductive episodes are not isolated but rather permeate the whole of modern mathematics—for Zermelo's type of justification "is inherited by those of us who come after him." It seems but a small step from this observation to the view that an adequate account of the development of mathematics would do well to focus not on the 'epistemic aims' of investigators but rather on the retroductive or abductive aims of practicing mathematicians.

MATHEMATICAL INTUITIONS AND PERCEPTION

As Kitcher maintains, mathematicians inherit problem-situations from past and present authoritative practitioners and on the whole direct their energies toward developing further whatever ideas strike them as most promising. An outstanding example is Kurt Gödel. Toward the end of a survey of what he takes to be a thoroughly unsatisfactory state of affairs in set theory, his mental image of a set induces him to claim that it is "perfectly possible" that a new set theory can be developed which will be able to resolve some important questions which are not merely technical since the

[8] Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, vol. 1, 2d. ed. (Cambridge: Cambridge University Press, 1963), p. v.

paradoxes of set theory pose serious problems both for logic and for epistemology.[9] Specifically, Gödel envisages the discovery of a new axiom of set theory that would be powerful enough to lead to a decision as to the truth or falsity of Cantor's continuum hypothesis.[10]

After rehearsing various approaches to this problem, and the different ways in which the meanings of such a pivotal, indispensable idea as that of a set can be analyzed, Gödel concludes that there is no reason not to think that a new mathematical intuition may eventually resolve the issue. He notes that no collection of axioms at the base of set theory forms a closed system that cannot be extended by the inclusion of new axioms. Although set-theoretical concepts are remote from sense experience, nothing stands in the way of thinking that there may still be something like a perception of the essence of set theory. This is because some of "the axioms force themselves on us as being true" (CCP, 268). This partly empiricist, partly platonistic assertion is particularly noteworthy because a mathematical intuition, Gödel observes, is no more or less hard to understand than an empirical observation in physics: "the question of the objective existence of the objects of mathematical intuition...is an exact replica of the question of the objective existence of the outer world" (CCP, 268). That is to say, mathematical ideas can be grasped in a manner similar to the way empirical ideas are formed—by an operation of thought performed on an "immediately given." In the case of mathematics this 'given' is distinguishable from yet analogous to the 'givens' that are involved in sense-experiences.

In the latter context, Gödel alludes to what I am claiming is an, if not *the*, *ur*-problem of naturalism—how to conceive the relationship between knowers and known —for he indicates the necessity of adopting a creative-constructive view of perception. It is thus worth quoting him more fully:

> it by no means follows … that [abstract ideas, such as that of object itself] … because they cannot be associated with actions of certain things upon our sense organs are purely subjective as Kant asserted. Rather they, too, may represent an aspect of objective reality, but, as opposed to sensations, their presence in us may be due to another kind of relationship between ourselves and reality (CCP, 268).

In other words, Gödel's struggles with the continuum hypothesis and the notion of mathematical intuition lead him to the heart of the fundamental philosophical problem of perception, one that, as Gödel rightly suggests, Kant exposed but did not resolve when he proposed, in effect, that experience *tout court* depended on a synthesizing faculty capable of generating "unities out of manifolds."[11]If this is so, experience is

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[9] Kurt Gödel, 'What Is Cantor's Continuum Problem?', in *Kurt Gödel: Collected Works*, volume II, ed. Solomon Feferman et al (New York: Oxford University Press, 1990), pp. 267-69 (hereafter referred to as CCP).

[10] A statement of this hypothesis is: any infinite subset of the real number continuum has the power of the set of integers or of the whole continuum.

[11] Perception in general is thus a world-making activity that can produce (in Gödel's words) "*one* object out its various aspects" (Ibid.)

anything but "kindly" in the sense envisioned by modern empiricists who yearn for a knowledge based on clear and definite observations grounded in sense-based 'facts'. By comparing the 'givens' of certain mathematical intuitions with the 'givens' presupposed by empirical observations, Gödel is in effect urging that the meaning of empirical knowledge must be stretched to include intuitive knowledge. At the same time he implicitly suggests that when the 'givens' responsible for mathematical intuitions result in systems of symbolisms that point to the existence of a connection between mathematics and physical reality.

MATHEMATICAL NATURALISM WITH INTUITIONS

I am maintaining that a full-bodied mathematical naturalism must begin by accepting the existence of genuine in-sights into reality; that intuitions, some of them anyway, deserve to be included in the class of natural objects. Efficacious symbolisms, in other words, may owe most of their powers to obscure operations of thought which, in an evolutionary world, perhaps only yield their treasures to the light of consciousness gradually and hesitantly. Indeed, the implication that consciousness emerges from unconsciousness is implicit in the very idea of evolution, which indicates in turn that the burning question of naturalism is what *kind* of story-telling could do justice to a world which includes a great variety of different sensibilities some of which have fashioned systems of symbolism that yield a kind of access to Nature.

I have suggested that Gödel points up the real difficulties in framing a naturalistic account of mathematics. For even if Kant is right and understanding is amenable to being explained in terms of categorial schemes, the puzzle of their genesis still remains. Gödel's own investigations into the formal symbolisms of mathematics indicate that mathematics is very far from being able to provide an answer, since it is unable to show that it is itself an unproblematic repository of apodictic truths. His incompleteness theorems, in particular, show that if objective truth is somehow embodied in specific mathematical systems, it is a kind of truth that cannot be pinned down precisely since (at least in the case of fairly unsophisticated systems) it belongs to the whole, inherently open-ended system.

It is also worth noting that since Gödel's results are theorems in the theory of formal systems, his work nicely illustrates Peirce's insistence on the hypothetical character of all scientific theorizing. As he observes, in mathematics one "merely posits hypotheses, and traces out their consequences" (CP, 1.240).[12] Indeed, how could the symbolisms of mathematics encode anything except 'conditional necessities' if their genesis and development depends on retroductive moments of mind whose established results (as Kitcher rightly notes) permeate all subsequent reasonings? That this sort of hypothesizing is moreover highly dependent on imagination is no news to practicing

[12] Or again, mathematics is "preeminently a science that reasons," yet it produces "nothing but conditional propositions." Many of Peirce's views on mathematics are collected in the chapter entitled 'The Nature of Mathematics,' in Buchler, op. cit.; see esp. pp. 142-3.

mathematicians. They regularly lend support to Peirce's observation that the mathematician "makes constructions in the imagination according to abstract precepts, and then observes these imaginary objects, finding in them relations of parts not specified in the precept of construction" (CP, 1.240). This suggests that the notion of mathematical intuition might better be called, instead of 'real in-sight' or 'perspicuous seeing', a 'perspicacious imagining'. And that if and when mathematical symbolisms are linked to empirical observations, or vice versa, what is going on has more affinity with artistic creation than with scientific investigation (as this is currently conceived).

Hence the would-be mathematical naturalist might well take note of Peirce's observation, that it takes poetic genius to create hypotheses (CP, 4.238). Or to put this another way, the chief difference between the retroductive creations of mathematicians and poets perhaps derives from the fact that they dwell at opposite ends of a linguistic spectrum, for the chief function of any language could be, apart from practical uses, to provide a means to express and communicate the insights and intuitions produced by perspicacious imaginations. And just as some poets and novelists conjure up possible worlds using the tropic resources of whatever natural language(s) they feel most at home in, so creative mathematicians explore the recesses of mathematical symbolisms in the hope of turning up new and significant patterns. The quality of the metaphorical and/or analogical devices that facilitate both kinds of hypothesizing is thus not incidental to whatever success is achieved; indeed, the question of quality is perhaps central since the emergence of significant additions to extant symbolisms may well depend on *how* well they can accommodate the powers inherent in imagination—a consideration borne out by the fact that popular metaphors and analogies can be good or bad, enlightening or misleading.

In any case, it is no more reprehensible or irrational for naturalistic philosophers of mathematics to enlist poetic metaphors than it is for poet-philosophers to enlist systematic mathematical or physical analogies.[13] The best interpretations of this world inspired by 'exact' science probably arise from ingenious feats of systematic 'metaphoring' whose successes attest not only to a special intellectual training but also an imaginative/intuitive talent for discerning the relevance of certain kinds of mathematical patterns. Hence a naturalistic account of mathematical knowledge cannot by-pass the socio-cultural dimension of knowledge-making, which indicates that mathematics ought generally to be regarded as a repository of more or less efficacious symbolisms that, like those of natural languages, provide a reservoir of systematic analogies which function like word-metaphors; that is, carry elements of meaning across the gaps whose presence is indicated by the inherent vagueness and ambiguities

[13] That this point needs constant reiteration is one of the main lessons of the recent 'Sokal affair', which hinges on the charge by apologists of science that 'post-modern' or culturally-oriented philosophers are guilty of intellectual laxity on account of their tendency to use (and usually abuse) technical metaphors or analogies drawn from the latest theories of mathematics and physics, which they do not fully understand. Yet it can be argued that all philosophers resort to imagery that requires interpretation and which may or may not further clarity of understanding.

of ordinary word-symbols.

MATHEMATICS AND LANGUAGE

As the historian Salomon Bochner argues, the best and most enduring parts of mathematics appear to be those that have been pursued for their own sake, rather than for some practical purpose. Claiming furthermore that 'prefabricated' mathematics, or mathematics for its own sake, has a certain "intellectual self-sufficiency,"[14] he argues that mathematics deals "with objects of its own 'aesthetic' perception and aprioristic emanation [which are] internally conceived, internally created, and inwardly structured…". Thus evoking an affinity between mathematics and creative art, he stresses that mathematics illustrates a "great power of creativity which resides within its compass" (RM, 47). At the same time Bochner appears to allude to powers that recall those which Descartes invokes; that is, powers related to an essentially spiritual capacity to symbolize, a capacity which is however evolutionary. For the efficacy of modern mathematical symbolisms is of quite a different order from that of the symbolisms used by the Greeks. The latter confined their use of mathematical symbols to what Bochner calls first order abstractions which express 'intuitive' or direct "idealizations from immediate actuality and 'external' reality. In other words, the Greeks "remained within the purview of what is called 'intuitive' in an obvious and direct sense"—in sharp contrast, with the makers of modern mathematical symbolisms who encode higher order abstractions, or "abstractions from abstractions" or "abstractions from intellectually conceived possibilities and potentialities " (RM, 51).

But if mathematics has now evolved to a point where it is chiefly concerned with abstractions of the 'second order', a principle worry for philosophers of mathematics is how this sort of intellectual activity (which concerns itself only with such 'internal' matters as formal consistency, aesthetic value, etc.) could come up with 'second order' abstractions having the power to organize thought about natural orderings in, for instance, the mysterious quantum realm—features of which might not have been suspected of existing were it not for the assistance of certain highly sophisticated mathematical theories.

Bochner himself does not explore the profound implications of his own line of thought, however. But in bringing out the importance of thinking about the peculiar character of mathematical symbolisms which illustrate different orders of abstraction he merely lends support to the view that mathematics may best be viewed as but one member of the panoply of human languages; perhaps lying at one far end of the spectrum of significant systems of symbolism. For mathematics would seem to be unique in the sense that it is an entirely written language on account of the highly

[14] Salomon Bochner, *The Role of Mathematics in the Rise of Science* (Princeton: Princeton University Press, 1966), 42 (hereafter referred to as RM). This feature, as Bochner notes, generates the particularly sticky problem of "the validity and significance of mathematical knowledge for other knowledge, especially for scientific knowledge" (RM, 46).

abstract meanings attached to its symbols.[15]

To round off this discussion, then, it may be useful to consider briefly Noam Chomsky's reflections on the underpinnings of language. Recalling some of the utterances of Descartes which I quoted earlier, Chomsky also appears to allude to certain hidden powers in human minds when he maintains that there is a linguistic faculty which is "a common human possession." If such a faculty does indeed exist, it does not seem to be a big step to the parallel claim that there is also a mathematical faculty which, like the linguistic faculty, is not given ready formed at birth but needs to be nurtured and developed. That is to say, every competent language user manifestly possesses a capacity to manipulate abstract notions by means of word-symbols and this capacity seems not unlike that exemplified by mathematicians. Perhaps all that distinguishes the mathematical faculty from the linguistic faculty is that the former is particularly well cultivated by those types of thinkers who, for whatever reason, have a special interest in abstract patterns, some (but not all) of which may be able to express the connections that obtain within and between natural events.

Chomsky in fact holds that the linguistic faculty involves "principles or notions implanted in the mind" which are a "direct gift" from nature. (ND,186). Being a component of human brains, this faculty develops from an initial state which is so similar across the human species that "we can reasonably abstract to *the* initial state of the language faculty."[16]But if this is so, and assuming that human brains are part of Nature, why not think that a similar claim applies to the mathematical faculty, or any other faculty for that matter (for the door is now open to think about "operations of thought" in terms of operations of faculties)?

This way of looking at mathematics is compatible, at least up to a point, with Chomsky's espousal of 'methodological naturalism' which he explicitly opposes to the 'methodological dualism' that informs the thinking of contemporary naturalists who tend to separate the world into two disjoint realms; one inhabited by 'thinking things' (and language users) and the other by 'things thought about'. Explicitly denying this type of dualism, which presupposes distinct boundaries between the mental and the physical, Chomsky therefore lays a ground for a thoroughly non-Cartesian version of naturalism which views language as much *in* the world as the mindful creatures who use it. Indeed, he refers to both language and mind as natural objects. The language faculty is a natural product of evolution which, during the course of its developments in individual human beings, is subject to 'local' conditions which are themselves influenced by the culture in which the individual is embedded.

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[15] This view is explicit in, for instance, Whitehead's approach to mathematics: he notes, for instance, that Algebra "is essentially a written language, and it endeavours to exemplify in its written structures the patterns which it is its purpose to convey" (Alfred North Whitehead, *Essays in Science and Philosophy*, New York: Greenwood Press, 1968, 107).

[16] Noam Chomsky, "Naturalism and Dualism in the Study of Language and Mind" (*International Journal of Philosophical Studies* Vol. 2(2), 181-209), 183 (hereafter referred to as ND).

As for *how* the linguistic faculty develops, Chomsky holds that "the environment triggers and to a limited extent shapes an internally directed process of growth, which stabilizes (pretty much) at about puberty" (ND, 183). So it is also worth stressing that for him the linguistic faculty is not independent of other faculties, for there is also a "science-forming faculty"; that is, a problem-oriented faculty since it is capable (presumably if properly developed) of identifying "problem situations" (which Chomsky identifies as certain cognitive states of belief, understanding, or misunderstanding), that guide in turn the formation of questions and thoughts about how they can be answered (or perhaps reformulated), what measures of experimental or empirical testing of results may work, and so on (ND, 188). His observations thus recall Peirce's emphasis on the thoroughly abductive nature of scientific inquiry which I have earlier argued depends on what may be even more fundamental faculties than the ones named above.

In any case, it seems highly significant that Chomsky defends his position in a manner that resonates with Peirce's Critical Commonsensism. He suggests that the underpinnings of rational inquiry consist of extremely vague, fundamental assumptions. That is to say, a naturalistic theory of mind and language can properly presuppose the truth of certain common sense observations. For there is nothing whatever to prevent a naturalist of any stripe whatever from boldly asserting that there are mental aspects of worldly events that are as 'real' as certain physical aspects. As Chomsky rightly points out, none demands that the true criteria for applying the terms 'electrical' or 'chemical' in descriptions of physical events be spelled out before physicists launch detailed inquiries into the electro-chemical secrets of matter. Likewise, it would be just as unreasonable to demand that the mental furniture of the world be clearly and precisely determined in advance of a naturalistic inquiry into the natural phenomena referred to by the terms 'mind' and 'language'.

Hence one of the great merits of Chomsky's naturalistic approach to language is that he brings out the principal difficulty in naturalistic explanation, which is surely where and how to begin. However, although he explicitly acknowledges that "there are interesting questions as to how naturalistic inquiry should proceed" (ND, 182), he does not take this half of the *ur*-problem of naturalism very seriously. Quite the contrary, he merely reiterates the credo of most contemporary naturalists: one may "simply adopt the standard outlook of modern science" (ND, 182), while at the same time presupposing that this outlook embodies the "normal canons of inquiry." The upshot is that Chomsky does not just "loosely" and uncontroversially associate mind with brains (a naturalist could hardly deny that brains have something to do with conceptual activity); he in fact ties mind very tightly to material states of the brain, a move that is so typical of contemporary naturalisms that one may fairly ask whether his approach is really all that different from the dualistic approaches he is opposing.

Why think that mind and language are natural objects on a par with those studied by, say, chemists or physicists? Chomsky in fact acknowledges in more than one place that materialistic or physicalist theories have failed to demonstrate this crucial point. Speaking of the common ground between naturalistic inquiry, as he understands it, and the deliverances of the successful natural sciences, he suggests in fact that their intersection could be empty.[17] Again, he observes that "the reach of naturalistic inquiry may be quite limited, not approaching questions of serious human concern, however far-reaching its intellectual interest may prove to be. That is surely the present condition, and might so remain" (ND, 194). Thus Chomsky's account of what he takes to be the most promising (in the sense of lacking any reasonable alternative) "intellectual" approach to language and mind seems no more compelling than Descartes' recommendations about how to begin to do philosophy.

It may be doubted, in short, if much enlightenment in respect to the relations between mind and language can be expected from a theory which revolves about the technical notion of an initial state. Chomsky argues that one must translate, for example, informal locutions such as: "Jones knows (speaks, understands, has) English", into a statement expressing the connection between a cognitive state of Jones' brain and a state of the world. The former state underlies Jones' "knowledge of many particular things: his knowing how to interpret linguistic signals, or that certain expressions mean what they do, and so on" (ND, 186). But it is just at this crucial point, where the notion of interpretation is slipped into a story about formal links (or mappings) between brain-states, regarded as complex systems with properties, and analogous states in other brains or in the world, that doubts become warranted. To view the interpretation of signs as an activity that is amenable to systematic 'mappings' between minds and world is to overlook the especial vagueness of word-signs, for the interpretation of linguistic signals more often than not calls for a kind of imaginative participation in a whole context and so is inherently error-prone. Interpretation, in other words, is arguably an essentially intuitive/imaginative process that simply cannot be reduced to "empirical hypotheses about biological endowment, interactions with the environment, the nature of the states attained, and their interactions with other systems of the mind (articulatory, perceptual, conceptual, intentional, etc.)" (ND, 186-87).

On the other hand, and in keeping with the view that the linguistic faculty is not given ready-formed, the understanding of vague word-signs often bears witness to the conjoint operation of a number of faculties, some of which may be only more or less well-developed (as when a particularly subtle description of a landscape is grasped by a colour-blind listener), so that interpretation conceivably requires the cooperative efforts of whole families of faculties which develop not from an initial state so much as from a condition of latency.

In other words, once one brings forth the idea of a linguistic faculty, one is on the way to calling for a very broad theory of representing that is capable of dealing with the possibility of many types of hidden "operations of thought," some of which may never become fully or properly developed. Hence, if human language is indeed a natural endowment of an evolving Nature (that is, the result of an emergent linguistic faculty), one must sooner rather than later contemplate the possibility of a primordial reality-

[17] "It is unknown whether aspects of the theory of mind—say, questions about consciousness—are problems or mysteries for humans" (ND, 188).

producing faculty (as Gödel indicates) which is presupposed by every attempt to account for effective systems of symbolism. Beginning thus, the naturalist can at least hope to do justice to all those aspects of experiencing that are mediated by signs and symbols and which cannot be directly related to the deliverances of the senses. The price is of course a final relinguishment of the Cartesian dream of secure and certain knowledge. One can only hope to come up with a more or less plausible and comprehensive story about the relations between minds and Nature, which is the sort of hope that both Kitcher and Chomsky in fact allude to. Kitcher observes, for instance, that "in both ideal gas theory and in mathematics, we tell stories—stories designed to highlight salient aspects of a messy reality" (MN, 324). Chomsky goes further and observes that by reading novels and studying history "we learn much more of human interest about how people think and feel and act...than from all of naturalistic psychology, and perhaps always will" (ND, 183).

But if all that reading novels can provide a willing reader are moments of insight that presuppose the existence of an already cultivated imagination, the question arises whether everything depends on a prior period of cultivation of imagination which is hardly simple since it involves the intersection of nature and culture. That is to say, an undivided nature-culture perhaps ought to become the principal arena in the search for any sort of naturalistic epistemology. Neither Kitcher nor Chomsky seem prepared to go this far, however. Chomsky maintains rather that "the place to look for answers [about problems connected with understanding mind and language] is where they are likely to be found: in the hard sciences, where richness and depth of understanding provides some hope of gaining insight into the questions" (ND, 182-83). He thereby adds fuel to the same suspicion that Kitcher engenders—that much of the thinking about mind and language of contemporary naturalists is vitiated by an acritical, if not irrational, belief in the paradigmatic rationality of science.

ON NATURALISM WITHOUT SCIENCE

Yet Chomsky's postulate of a language faculty could well serve as a point of departure for a comprehensive naturalism which, notwithstanding his desire to avoid "metaphysical connotations" (ND, 182), nonetheless inevitably raises profound metaphysical questions. The nature of the difficulties are hinted at in the 'truisms' that one cannot help but appeal to in order to say anything intelligible at all. That is to say, in the context of the quest for a true mathematical naturalism, everything probably depends on what intuitions or insights one is inclined to puts one's trust in.

Despite the likely objection that the symbolisms of mathematics are nonetheless privileged and need to be distinguished sharply from the symbolisms of natural languages, since their encodings of meaning are guided by the self-evident and stable rules of logic, nothing in fact stands in the way of thinking that mathematics is like every other kind of symbolism that is under the influence of cultural forces. Indeed, the different ways in which mathematics has been developed in different cultures ought long ago to have put paid to the Cartesian vision of mathematics as the repository of certain knowledge. For the mathematical systems that come to dominate in a given culture very likely reflect the peculiar interests of its members.[18] Such a possibility is fully in accord with the now widely accepted fact that our ways of looking at and acting in the world are subject to ever-changing conditions, so that to ignore the evolutionary factor in the making of mathematical theories is not only to risk losing all touch with reality, it is to risk forfeiting all hope of making sense of mathematics.

It is by now almost a commonplace that learning how to find one's way about in any kind of symbolism is like learning how to play a game, where some of the rules are spelled out initially and some have to be divined as one goes along. So all that seems possible to assert with any degree of confidence is that the sort of mathematical symbolisms that mathematicians in the West have learned how to develop have actually helped throw some light on how the ordering exhibited by certain physical events can be expressed. Yet if the world is best viewed not as a static Universe but rather as an evolving complex of different forms of life and hence modes of experiencing, any form of symbolism may well have an evolutionary character that mirrors the human animal's evolving efforts to make meaning; that is, to organize and reorganize its experiences through evolving new meanings from extant meanings.

The history of mathematics appears to bear this out. For the advance of mathematical theory proceeds not from the more primitive or elementary states of systems (e.g., projective geometry) to the more sophisticated and complex systems (e.g., Euclidean geometry) but rather the other way round. This situation accords with the general implication of evolution which is that consciousness has emerged from unconsciousness and thus (assuming that the notions of emergence and intuition/imagination are as intimately connected as I am maintaining) that mathematical intuitions are just as emergent as other forms of intuition. Hence it is essential for the naturalist to address this question of questions almost immediately, which is hardly a simple matter if the evolution of consciousness generally alludes to changing ways of harmonizing 'inner' processes and 'outer' processes, assuming that harmony is an apt metaphor for conveying the meaning of good thinking. But since it is hardly clear what the latter notion means, it is perhaps better to say that one of the more urgent tasks of philosophy *tout court*, and not just mathematical naturalism, is to begin by trying to frame a plausible account of how meaning is actually made in this evolutionary world, on the understanding that what has been aptly called (by Wigner) the "remarkable efficacy" of mathematics may supply a good reason for believing in the occasional emergence of insightful (or intuitively grasped) 'real' meanings.

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[18] That the logic of any sort of mathematical language can be regarded as mirroring the logic(s) that happen to prevail in the dominant culture—logics that can be discerned in peculiar speech patterns and which very likely mirror different ways of world-making—is argued by Helen Verran, *Science and an African Logic* (Chicago: University of Chicago Press, 2001). Maintaining that worlds are generated in acts, she holds that the collective enacting of their inhabitants, who are as likely to be non-human as human, is at once material and symbolic.